The above results generalize theorems of Nelson[5] and Nelson and Stinespring.[1]

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[1] Nelson, E., and W. F. Stinespring, *Amer. J. Math.*, **81**, 547–560 (1959).

[2] Langlands, R. P., *On Lie Semi-Groups* (submitted to *Canad. J. Math.*).

[3] For terminology see Hille, E., and R. S. Phillips, *Functional Analysis and Semi-Groups*.

[4] These are essentially estimates for the resolvents of strongly elliptic differential operators with constant coefficients and are obtained from estimates for the fundamental solution of the associated parabolic equation similar to those of Silov G. E., (*Usp. Mat. Nauk.*, **10**, 89–100 (1955)).

[5] *Ann. Math.*, **70**, 572–615 (1959).

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*THE EXTENT OF ASYMPTOTIC STABILITY**

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In studying the stability of a system it is never completely satisfactory to know only that an equilibrium state is asymptotically stable or for that matter to know only that it is unstable. In a mathematical sense it may be asymptotically stable but from a practical point of view be unstable, and, conversely, it may be mathematically unstable but practically stable. Both stability and instability are local concepts. An equilibrium state of a system may be unstable, and yet it may be true that the system always tends to return, perhaps not to the equilibrium state itself, but sufficiently near the equilibrium state. An equilibrium state can be asymptotically stable and yet perturbations which would be considered small compared to the perturbations to be expected may cause the system to drift away from the equilibrium never to return. As a practical matter, it is necessary to have some idea of the size of the region of asymptotic stability. It is never possible to do this by examining only the linear approximation to the system. The effect of nonlinearities must be taken into account, and Liapunov's stability method[1] provides a means of doing this. The purpose of this paper is to report some mathematical theorems that underlie methods for estimating regions of asymptotic stability. These methods, with examples illustrating them, are to be discussed elsewhere in greater detail.[2]

The system whose stability is being investigated is described by the vector differential equation

$$\dot{x} = X(x). \tag{1}$$

The state of the system at time t is an n-vector $x(t) = (x_1(t),\ldots,x_n(t))$. The phase velocity $dx/dt = \dot{x}$ is defined by the vector field $X(x) = (X_1(x),\ldots,X_n(x))$. For each initial state x° we assume there is a unique solution $x(t)$ of (1) satisfying $x(0) = x^\circ$, and that this solution depends continuously on the initial state x°. The equilibrium state being investigated is at the origin: $X(0) = 0$. The Liapunov method depends upon the construction of a suitable domain Ω and a suitable

Liapunov function $V(x)$, which is a kind of generalized energy function. We assume throughout that $V(x)$ has continuous first partials in Ω. With reference to the system (1), we define

$$\dot{V}(x) = \frac{\partial V}{\partial x_1} X_1 + \ldots + \frac{\partial V}{\partial x_n} X_n = (\text{grad } V) \cdot X.$$

If $x(t)$ is a solution of (1), then

$$\frac{d}{dt} V(x(t)) = \dot{V}(x(t)).$$

No knowledge of the solutions of (1) is required to compute $\dot{V}(x)$. It is computed directly from a knowledge of the differential equations which describe how the system changes. The difficulty lies in constructing $V(x)$. This requires experience and technique—a technique in which Russian mathematicians and engineers excel. The theorems stated below describe what are suitable Liapunov functions for determining the extent of asymptotic stability. The basic result is

THEOREM 1. *Let Ω be a bounded closed set with the property that every solution of* (1) *starting in Ω remains for all future time in Ω. Suppose there is a scalar function $V(x)$ with the property that $\dot{V}(x) \leqq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.*

In some instances the construction of a Liapunov function $V(x)$ will itself guarantee the existence of a set Ω. For instance,

THEOREM 2. *Let Ω denote the closed region defined by $V(x) \leqq l$. If, in addition Ω is bounded and $\dot{V}(x) \leqq 0$ in Ω, then every solution starting in Ω approaches M as $t \to \infty$. (M is set defined in Theorem 1.)*

Note with regard to Theorem 2 that if $V(x) \to \infty$ as $||x|| \to \infty$, then the set Ω defined by $V(x) \leqq l$ is bounded for all values of l. If $\liminf_{||x|| \to \infty} V(x) = l_0$, then Ω is bounded for all $l < l_0$.

Thus, under suitable circumstances, the set Ω is an estimate of the region of asymptotic stability. According to Theorem 1 the procedure is to find a region Ω and a suitable function $V(x)$. If $\dot{V}$ does not vanish identically along any solution starting in Ω except the origin, then every solution in Ω approaches the origin at $t \to \infty$. It seems in some examples to be easier to separate the problems of finding Ω and constructing a Liapunov function $V(x)$ although as Theorem 2 points out the Liapunov function itself may determine Ω.

If the origin is stable and every solution approaches the origin as $t \to \infty$, then the system is said to be *completely stable* (asymptotically stable in the large). The region of asymptotic stability is the whole space. The basic theorem leading to methods for establishing complete stability is

THEOREM 3. *Let $V(x)$ be a scalar function with continuous first partials for all x. Assume that*

i. $V(x) > 0$ *for all* $x \neq 0$
ii. $\dot{V}(x) \leqq 0$ *for all* x.

Let E be the set of all points where $\dot{V}(x) = 0$ and let M be the largest invariant set contained in E. Then every solution bounded for $t > 0$ approaches M as $t \to \infty$.

To establish the complete stability of a system one needs (i) to establish that all solutions are bounded for $t \geqq 0$ and (ii) to construct a function $V(x)$ satisfying the conditions of Theorem 3 and such that M is the origin. Here again one may be able to conclude from the Liapunov function $V(x)$ itself that all solutions are bounded for $t \geqq 0$. This is true, for instance, if $V(x) \rightarrow \infty$ as $||x|| \rightarrow \infty$, although it often is easier to consider (i) and (ii) as separate problems. Boundedness is a type of stability and can itself be investigated by Liapunov methods.[3]

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[1] Among the available references on Liapunov's method may be mentioned, (*a*) Hahn, W., *Theorie und Anwendung der Direckten Methode von Ljapunov* (Berlin: Springer-Verlag, 1959). (*b*) Antosiewicz, H. A., "A survey of Liapunoc's second method," in *Contributions to the Theory of Nonlinear Oscillations IV*, Annals of Math. Studies No. 41 (Princeton University Press, 1958). (*c*) Cesari, L., *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations* (Berlin: Springer-Verlag, 1959). (*d*) Malkin, I. G., *Theory of Stability of Motion*, AEC Translation Series, AEC-t-3352 (translated from a publication of the State Publishing House of Technical-Theoretical Literature, Moscow-Leningrad, 1952).

[2] In a paper by the author to appear in the latter part of 1960 in a Special Nonlinear Issue of the *Proc. of the IRE.*

[3] This has been studied extensively by Taro Yoshizawa. See his paper on "Liapunov's Function and Boundedness of Solutions," *Funkcialaj Ekvacioj* **2**, 95–142 (1959).

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*VIRUS-CELL INTERACTION WITH A TUMOR-PRODUCING VIRUS**

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The polyoma(PY) virus or parotid tumor agent[1, 2]—a DNA-containing virus[3, 4]—is characterized by a duality of action: it produces neoplasias of various types in different species of rodents,[5] and causes cell degeneration in mouse embryo tissue cultures.[6] In the experiments to be reported here, it was possible to obtain in cellular cultures *in vitro* the oncogenic effect of the virus; this afforded the possibility of studying the relationship between the oncogenic and cytocidal effect of the virus. The results so far obtained reveal a situation novel in animal viruses and suggest the existence of a host-virus interaction with characteristics reminiscent of temperate bacteriophage.

Material and Methods.—The PY virus was obtained from Dr. Rowe of the National Institutes of Health. A stock was prepared from a single plaque and serial passages of this stock in mouse embryo tissue cultures were used for the experiments. In these passages, the virus maintained both the cytocidal activity in mouse embryo tissue cultures and the property of eliciting heart, liver, and kidney sarcomas within a few weeks after injection into newborn Golden hamsters. The virus was assayed by plaque formation on mouse embryo monolayer cultures by means of the technique previously described.[7]